Combinatorial Structure of Manifolds with Poincaré Conjecture

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Abstract: A manifold M^n inherits a labeled n-dimensional graph $\widetilde{M}[G^L]$ structure consisting of its charts. This structure enables one to characterize fundamental groups of manifolds, classify those of locally compact manifolds with finite non-homotopic loops by that of labeled graphs G^L. As a by-product, this approach also concludes that *every homotopy n-sphere is homeomorphic to the sphere S^n for an integer $n \geq 1$*, particularly, the Perelman's result for $n = 3$.

Key Words: Combinatorial Euclidean space, manifold, fundamental group, labeled graph, homotopic loop.

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§1. Introduction

An n-manifold is a second countable Hausdorff space of locally Euclidean n-space without boundary. Terminologies and notions used in this paper are standard. We follow references [1], [13] for topology, [5], [6] for graphs or topological graphs and [11] for combinatorial manifolds. Here, we only mention conceptions appeared in references [8], [10]-[11] for combinatorial manifolds.

Definition 1.1 *A combinatorial Euclidean space $\mathscr{E}_G(n_\nu; \nu \in \Lambda)$ underlying a connected graph G is a topological spaces consisting of $\mathbf{R}^{n_\nu}$, $\nu \in \Lambda$ for an index set Λ such that*

$$V(G) = \{\mathbf{R}^{n_\nu} | \nu \in \Lambda\};$$

$$E(G) = \{\ (\mathbf{R}^{n_\mu}, \mathbf{R}^{n_\nu}) |\ \mathbf{R}^{n_\mu} \cap \mathbf{R}^{n_\nu} \neq \emptyset, \mu, \nu \in \Lambda\}.$$

If $|\Lambda| = 1$, i.e., the dimension of Euclidean spaces in $\mathscr{E}_G(n_\nu; \nu \in \Lambda)$ are all in the same n, the notation $\mathscr{E}_G(n_\nu; \nu \in \Lambda)$ is abbreviated to $\mathscr{E}_G(n,\ \nu \in \Lambda)$ for simplicity.

Notice that a Euclidean space $\mathbf{R}^n$ is an n-dimensional vector space with a normal basis $\overline{\epsilon}_1 = (1, 0, \cdots, 0)$, $\overline{\epsilon}_2 = (0, 1, 0 \cdots, 0)$, $\cdots$, $\overline{\epsilon}_n = (0, \cdots, 0, 1)$, namely, it has n orthogonal orientations. In Definition 1.1, we do not assume $\mathbf{R}^{n_\mu} \cap \mathbf{R}^{n_\nu} = \mathbf{R}^{\min\{n_\mu, n_\nu\}}$.

In fact, let $\mathcal{X}_{v_\mu}$ be the set of orthogonal orientations in $\mathbf{R}^{n_{v_\mu}}$, $\mu \in \Lambda$, respectively ([12]). Then

$$\mathbf{R}^{n_\mu} \bigcap \mathbf{R}^{n_\nu} = \mathcal{X}_{v_\mu} \bigcap \mathcal{X}_{v_\nu}.$$

A *combinatorial fan-space* $\widetilde{\mathbf{R}}(n_\nu; \nu \in \Lambda)$ is a combinatorial Euclidean space $\mathscr{E}_{K_{|\Lambda|}}(n_\nu; \nu \in \Lambda)$ of $\mathbf{R}^{n_\nu}$, $\nu \in \Lambda$ such that for any integers $\mu, \nu \in \Lambda$, $\mu \neq \nu$,

$$\mathbf{R}^{n_\mu} \bigcap \mathbf{R}^{n_\nu} = \bigcap_{\lambda \in \Lambda} \mathbf{R}^{n_\lambda}.$$

If $|\Lambda| = m < +\infty$, for $\forall p \in \widetilde{\mathbf{R}}(n_\nu; \nu \in \Lambda)$ we can present it by an $m \times n_m$ coordinate matrix $[\overline{x}]$ following with $x_{il} = \frac{x_l}{m}$ for $1 \le i \le m, 1 \le l \le \widehat{m}$, where $\widehat{m}$ is the number of orthogonal orientations in $\mathcal{X}_{v_\mu} \bigcap \mathcal{X}_{v_\nu}$,

$$[\overline{x}] = \begin{bmatrix} x_{11} & \cdots & x_{1\widehat{m}} & x_{1(\widehat{m})+1)} & \cdots & x_{1n_1} & \cdots & 0 \\ x_{21} & \cdots & x_{2\widehat{m}} & x_{2(\widehat{m}+1)} & \cdots & x_{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ x_{m1} & \cdots & x_{m\widehat{m}} & x_{m(\widehat{m}+1)} & \cdots & \cdots & x_{mn_m-1} & x_{mn_m} \end{bmatrix},$$

which enables us to generalize the conception of manifold to combinatorial manifold, a locally combinatorial Euclidean space.

Definition 1.2 *For a given integer sequence $0 < n_1 < n_2 < \cdots < n_m$, $m \ge 1$, a topological combinatorial manifold $\widetilde{M}$ is a Hausdorff space such that for any point $p \in \widetilde{M}$, there is a local chart (U_p, φ_p) of p, i.e., an open neighborhood U_p of p in $\widetilde{M}$ and a homoeomorphism $\varphi_p : U_p \to \widetilde{\mathbf{R}}(n_1(p), n_2(p), \cdots, n_{s(p)}(p)) = \bigcup_{i=1}^{s(p)} \mathbf{R}^{n_i(p)}$ with*

$$\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\} \text{ and}$$

$$\bigcup_{p \in \widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} = \{n_1, n_2, \cdots, n_m\},$$

denoted by $\widetilde{M}(n_1, n_2, \cdots, n_m)$ or $\widetilde{M}$ on the context and

$$\widetilde{\mathcal{A}} = \{(U_p, \varphi_p) | p \in \widetilde{M}(n_1, n_2, \cdots, n_m))\}$$

an atlas on $\widetilde{M}(n_1, n_2, \cdots, n_m)$.

A topological combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is finite if it is just combined by finite manifolds without one manifold contained in the union of others.

If $m = 1$, then $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is exactly the manifold M^{n_1} by definition. Furthermore, if these manifolds M_i, $1 \le i \le m$ in $\widetilde{M}(n_1, n_2, \cdots, n_m)$ are Euclidean spaces $\mathbf{R}^{n_i}$, $1 \le i \le m$, then $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is nothing but the combinatorial Euclidean space $\mathscr{E}_G(n_\nu; \nu \in \Lambda)$ with $\Lambda = \{1, 2, \cdots, m\}$.

For a finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ consisting of manifolds $M_i,\ 1 \leq i \leq m$, we can construct a vertex-edge labeled graph $G^L[\widetilde{M}]$ defined by

$$V(G^L[\widetilde{M}]) = \{M_1, M_2, \cdots, M_m\},$$

$$E(G^L[\widetilde{M}) = \{\ (M_i, M_j) \mid M_i \bigcap M_j \neq \emptyset, 1 \leq i, j \leq n\}$$

with a labeling mapping $\Theta : V(G^L[\widetilde{M}]) \cup E(G^L[\widetilde{M}]) \rightarrow \mathbf{Z}^+$ determined by

$$\Theta(M_i) = \dim M_i \ \text{ and } \ \Theta(M_i, M_j) = \dim M_i \bigcap M_j$$

for integers $1 \leq i, j \leq m$, which is inherent structure of combinatorial manifolds. Particularly, for a combinatorial Euclidean space $\mathscr{E}_G(n_1, \cdots, n_m)$, this vertex-edge labeled graph is defined on G by labeling $\Theta : V(G) \cup E(G) \rightarrow \mathbf{Z}^+$ determined by

$$\Theta(\mathbf{R}^{n_i}) = n_i \ \text{ and } \ \Theta(\mathbf{R}^{n_i}, \mathbf{R}^{n_j}) = \dim \mathbf{R}^{n_i} \bigcap \mathbf{R}^{n_j}$$

for integers $1 \leq i, j \leq m$.

The objective of this paper is to characterize the inherent combinatorial structure of n-manifolds with finite non-homotopic loops. For such manifolds, there is a well-known Poincaré conjecture first proved by Perelman ([3]) following.

Theorem 1.3(Perelman,[15]-[17]) *Any closed simply connected* 3*-manifold is homeomorphic to* S^3.

Notice that a *homotopy sphere* is an n-manifold homotopy equivalent to the n-sphere. A generalized Poincaré conjecture says that *any homotopy n-sphere is homeomorphic to S^n*. Combining works of Smale in 1961 for $n \geq 5$ ([18]), Freedman's in 1982 for $n = 4$ ([2]), Theorem 1.3 and classical result for $n = 1, 2$, we conclude that

Theorem 1.4 *Any homotopy n-sphere is homeomorphic to S^n for $n \geq 1$.*

Then *can we find a unified proof on Theorem* 1.4? By applying a combinatorial notion [9], we find an inherent labeled graph $G^L[M^n]$ structure consisting of charts homeomorphic to a Euclidean space $\mathbf{R}^n$ for a topological manifold M^n in this paper, which enables us to classify locally compact n-manifolds with finite non-homotopic loops by labeled graphs for integers $n \geq 1$, also simplifies the calculation of fundamental groups of n-manifolds and get a combinatorial proof for Theorem 1.4.

§2. Dimensional Graphs

We discuss combinatorial Euclidean spaces $\mathscr{E}_G(n)$ in this section whose importance is shown in the next result.

Theorem 2.1 *A locally compact n-manifold M^n is a combinatorial manifold $\widetilde{M}_G(n)$ homeomorphic to a Euclidean space $\mathscr{E}_{G'}(n)$ with countable graphs $G \cong G'$ inherent in M^n, denoted by $G[M^n]$.*

Proof Let M^n be a locally compact n-manifold with an atlas

$$\mathscr{A}[M^n] = \{ \ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda\},$$

where Λ is a countable set. Then each $U_\lambda, \ \lambda \in \Lambda$ is itself an n-manifold by definition. Define an underlying combinatorial structure G by

$$V(G) = \{U_\lambda | \lambda \in \Lambda\},$$
$$E(G) = \{ \ (U_\lambda, U_\iota)_i, 1 \leq i \leq \kappa_{\lambda\iota} + 1 | \ U_\lambda \bigcap U_\iota \neq \emptyset, \lambda, \iota \in \Lambda\}$$

where $\kappa_{\lambda\iota}$ is the number of non-homotopic loops in formed between U_λ and U_ι. Then we get a combinatorial manifold $\widetilde{M}_G(n)$ underlying a countable graph G.

Define a combinatorial Euclidean space $\mathscr{E}_{G'}(n, \lambda \in \Lambda)$ of spaces $\mathbf{R}^n$ by

$$V(G') = \{\varphi_\lambda(U_\lambda) | \lambda \in \Lambda\},$$
$$E(G') = \{ \ (\varphi_\lambda(U_\lambda), \varphi_\iota(U_\iota))_i, 1 \leq i \leq \kappa'_{\lambda\iota} + 1 | \ \varphi_\lambda(U_\lambda) \bigcap \varphi_\iota(U_\iota) \neq \emptyset, \lambda, \iota \in \Lambda\},$$

where $\kappa'_{\lambda\iota}$ is the number of non-homotopic loops in formed between $\varphi_\lambda(U_\lambda)$ and $\varphi_\iota(U_\iota)$. Notice that $\varphi_\lambda(U_\lambda) \bigcap \varphi_\iota(U_\iota) \neq \emptyset$ if and only if $U_\lambda \bigcap U_\iota \neq \emptyset$ and $\kappa_{\lambda\iota} = \kappa'_{\lambda\iota}$ for $\lambda, \iota \in \Lambda$. We know that $G \ \cong \ G'$ by definition.

Now we prove that $\widetilde{M}_G(n)$ is homeomorphic to $\mathscr{E}_{G'}(n, \lambda \in \Lambda)$. By assumption, M^n is locally compact. Whence, there exists a partition of unity $c_\lambda : U_\lambda \rightarrow \mathbf{R}^n$, $\lambda \in \Lambda$ on the atlas $\mathscr{A}[M^n]$. Let $A_\lambda = \mathrm{supp}(\varphi_\lambda)$. Define functions $h_\lambda : M^n \rightarrow \mathbf{R}^n$ and $\mathbf{H} : M^n \rightarrow \mathscr{E}_{G'}(n)$ by

$$h_\lambda(x) = \begin{cases} c_\lambda(x)\varphi_\lambda(x) & \text{if } x \in U_\lambda, \\ \mathbf{0} = (0, \cdots, 0) & \text{if } x \in U_\lambda - A_\lambda. \end{cases}$$

and

$$\mathbf{H} = \sum_{\lambda \in \Lambda} \varphi_\lambda c_\lambda, \quad \text{and} \quad \mathbf{J} = \sum_{\lambda \in \Lambda} c_\lambda^{-1} \varphi_\lambda^{-1}.$$

Then h_λ, $\mathbf{H}$ and $\mathbf{J}$ all are continuous by the continuity of φ_λ and c_λ for $\forall \lambda \in \Lambda$ on M^n. Notice that $c_\lambda^{-1}\varphi_\lambda^{-1}\varphi_\lambda c_\lambda$ =the unity function on M^n. We get that $\mathbf{J} = \mathbf{H}^{-1}$, i.e., $\mathbf{H}$ is a homeomorphism from M^n to $\mathscr{E}_{G'}(n, \lambda \in \Lambda)$. □

According to Theorem 2.1, a combinatorial Euclidean space homeomorphic to a n-manifold M^n can be denoted by $\mathscr{E}_{G[M^n]}(n, \ \mu \in \Lambda)$. We classify such combinatorial Euclidean spaces $\mathscr{E}_G(n, \mu \in \Lambda)$ into two classes by considering the intersection $\varphi_\mu^{-1}(\mathbf{R}^n) \cap \varphi_\nu^{-1}(\mathbf{R}^)$ for $\mu, \nu \in \Lambda$ following:

Class 1. For $\forall \mu, \nu \in \Lambda$, $\varphi_\mu^{-1}(\mathbf{R}^n) \cap \varphi_\nu^{-1}(\mathbf{R}^) = \emptyset$ or homeomorphic to $\mathbf{R}^n$.

Class 2. There are $\mu, \nu \in \Lambda$ such that $\varphi_\mu^{-1}(\mathbf{R}^n) \cap \varphi_\nu^{-1}(\mathbf{R}^) \neq \emptyset$ with more than 2 arcwise connected components.

We respectively discuss Classes 1 and 2 by dimensional graphs following.

2.1 **Dimensional Graphs $\widetilde{M}^n[G]$**

Let $B^n = \{(x_1, x_2, \cdots, x_n) | \ (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n, \ x_1^n + x_2^n + \cdots + x_n^n < 1\}$, $S^n = \{(x_1, x_2, \cdots, x_n) | \ (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n, \ x_1^n + x_2^n + \cdots + x_n^n = 1\}$ be a n-dimensional open ball or n-sphere for an integer $n \geq 1$, respectively. The combinatorial Euclidean spaces $\mathscr{E}_G(n, \lambda \in \Lambda)$ in Class 1 is such a topological space that each non-homotopic loops comes from the graph G, which can be characterized by graphs in spaces, i.e., *n-dimensional graphs* defined following.

Definition 2.2 *An n-dimensional graph $\widetilde{M}^n[G]$ is a combinatorial Euclidean space $\mathscr{E}_G(n)$ of $\mathbf{R}^n_\mu, \mu \in \Lambda$ underlying a combinatorial structure G such that*

(1) *$V(G)$ is discrete consisting of B^n, i.e., $\forall v \in V(G)$ is an open ball B^n_v;*

(2) *$\widetilde{M}^n[G] \setminus V(\widetilde{M}^n[G])$ is a disjoint union of open subsets $e_1, e_2, \cdots, e_m$, each of which is homeomorphic to an open ball B^n;*

(3) *the boundary $\overline{e}_i - e_i$ of e_i consists of one or two B^n and each pair $(\overline{e}_i, e_i)$ is homeomorphic to the pair $(\overline{B}^n, B^n)$;*

(4) *a subset $A \subset \widetilde{M}^n[G]$ is open if and only if $A \cap \overline{e}_i$ is open for $1 \leq i \leq m$.*

A *topological graph* $\mathscr{T}[G]$ of a graph G is a 1-dimensional graph in a topological space $\mathscr{P}$. We restate it in the following.

Definition 2.3 *A topological graph $\mathscr{T}[G]$ is a pair (X, X^0) of a Hausdorff space X with its a subset X^0 such that*

(1) *X^0 is discrete, closed subspaces of X;*

(2) *$X - X^0$ is a disjoint union of open subsets $e_1, e_2, \cdots, e_m$, each of which is homeomorphic to an open interval $(0, 1)$;*

(3) *the boundary $\overline{e}_i - e_i$ of e_i consists of one or two points. If $\overline{e}_i - e_i$ consists of two points, then $(\overline{e}_i, e_i)$ is homeomorphic to the pair $([0, 1], (0, 1))$; if $\overline{e}_i - e_i$ consists of one point, then $(\overline{e}_i, e_i)$ is homeomorphic to the pair $(S^1, S^1 - \{1\})$;*

(4) *a subset $A \subset \widetilde{T}[G]$ is open if and only if $A \cap \overline{e}_i$ is open for $1 \leq i \leq m$.*

Observation shows that there is a natural relation between a n-dimensional graph $\widetilde{M}^n[G]$ with that of its a topological graph $\mathscr{T}_0[G] = (X_0, X_0^0)$ which is constructed from $\widetilde{M}^n[G]_1$ by:

(1) Let $X_0^0 = \{ \ O_v \mid O_v$ is the center of B^n_v for $v \in V(G)\}$;

(2) For $\forall uv \in E(G)$, let uv be a line segment $e_{uv} : [0, 1] \to B^n_u \cup B^n_v$ such that $e_{uv}(0) = O_u$ and $e_{uv}(1) = O_v$.

Then, the following result shows that a n-dimensional graph $\widetilde{M}^n[G]$ is in fact a blown up of a topological graph $\mathscr{T}[G]$ to dimensional n.

Theorem 2.4 *For any integer $n \geq 1$, $\mathscr{T}_0[G]$ is a deformation retract of $\widetilde{M}^n[G]$.*

Proof If $n = 1$, then $\widetilde{M}^n[G] = \mathscr{T}_0[G]$ is itself a topological graph. So we assume $n \geq 2$. Define a mapping $f : \widetilde{M}^n[G] \times I \to \widetilde{M}^n[G]$ by

$$f(\overline{x}, t) = (1 - t)\overline{x} + t\overline{x}_0,$$

for $\forall \overline{x} \in \widetilde{M}^n[G]_1, t \in I$, where $\overline{x}_0 = O_v$ if $\overline{x} \in B_v^n$, and $\overline{x}_0 = p(\overline{x})$ if $\overline{x} \in e_i$, where $p : uv \to e_{uv}$ a projection for $1 \leq i \leq m$, such as those shown in Fig.2.1.

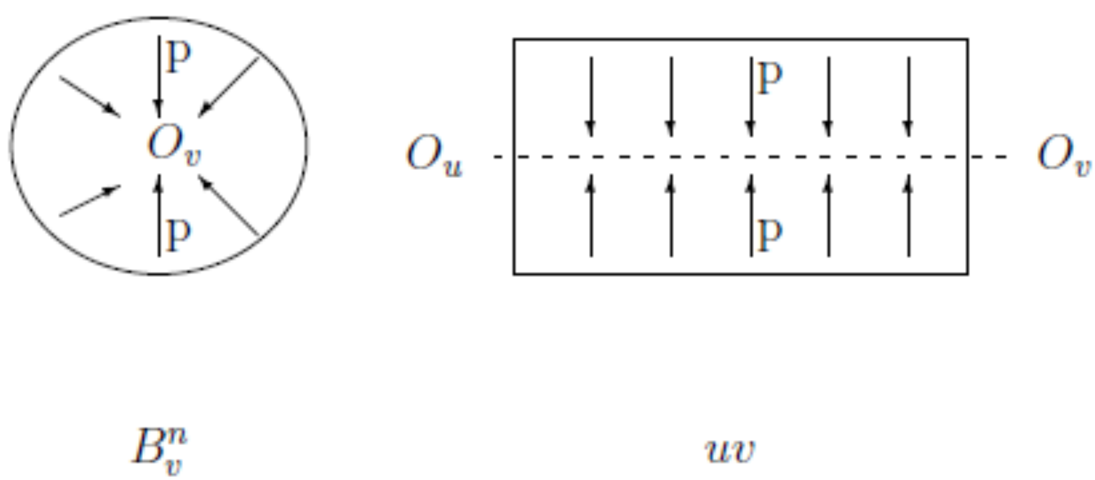

Fig.2.1

Then we know that f is continuous by definition and for $\forall \overline{x} \in \widetilde{M}^n[G]$,

$$f(\overline{x}, 0) = \overline{x},$$
$$f(\overline{x}, 1) = p(\overline{x}),$$

and $f(\overline{x}, t) = \overline{x}$ for $\forall \overline{x} \in \mathscr{T}_0[G]$ and $t \in I$. Therefore, $\mathscr{T}_0[G]$ is a deformation retract of $\widetilde{M}^n[G]$ by definition. $\square$

Notice that the inclusion mapping $i : \mathscr{T}_0[G] \to \widetilde{M}^n[G]$ is an isomorphism between groups $\pi(\mathscr{T}_0[G], v_0)$ and $\pi(\widetilde{M}^n[G]_1, v_0)$. We get a conclusion by Theorem 2.4 following.

Corollary 2.5 *Let $\widetilde{M}^n[G]$ be a n-dimensional graph. Then for $v_0 \in \mathscr{T}_0[G]$,*

$$\pi(\widetilde{M}^n[G], v_0) \cong \pi(\mathscr{T}_0[G], v_0).$$

We have known the structure of fundamental group of a topological graph $\mathscr{T}[G]$ in [13]. Whence, we can characterize the fundamental group of a n-dimensional graph $\widetilde{M}^n[G]$ by applying Corollary 2.5 following.

Theorem 2.6 *Let T_{span} be a spanning tree in the topological graph $\mathscr{T}_0[G]$, $\{e_\lambda : \lambda \in \Lambda\}$ the set of edges of $\mathscr{T}_0[G]$ not in T_{span} and $\alpha_\lambda = A_\lambda e_\lambda B_\lambda \in \pi(\mathscr{T}_0[G], v_0)$ a loop associated with $e_\lambda = a_\lambda b_\lambda$ for $\forall \lambda \in \Lambda$, where $v_0 \in \mathscr{T}_0[G]$ and A_λ, B_λ are unique paths from v_0 to a_λ or from b_λ to v_0 in T_{span}. Then*

$$\pi(\mathscr{T}_0[G], v_0) = \langle \alpha_\lambda | \lambda \in \Lambda \rangle .$$

An *n-dimensional tree* is a n-dimensional graph $\widetilde{M}^n[G]$ with a tree G, denoted by $\widetilde{M}^n[T]$. Applying Theorem 2.4, we know the next result.

Theorem 2.7 *An n-dimensional tree $\widetilde{M}^n[T]$ is contractible.*

Proof By Theorem 2.4, we know that there is a continuous mapping $f : \widetilde{M}^n[T] \times I \to \widetilde{M}^n[T]$ such that $\mathscr{T}_0[T]$ is a deformation retract of $\widetilde{M}^n[T]$, i.e., for $\forall \overline{x} \in \widetilde{M}^n[T]$,

$$f(\overline{x}, 0) = \overline{x},$$
$$f(\overline{x}, 1) = p(\overline{x}),$$

and $f(\overline{x}, t) = \overline{x}$ for $\forall \overline{x} \in \mathscr{T}_0[T]$ and $t \in I$. Notice that $\mathscr{T}_0[T]$ is contractible ([13]), we have a continuous mapping $g : \mathscr{T}_0[T] \times I \to \mathscr{T}_0[T]$ such that $\{v_0\}$ is a deformation retract for $\forall v_0 \in \mathscr{T}[T]$. Whence, the composition mapping $g \circ f : \widetilde{M}^n[T] \times I \to \widetilde{M}^n[T]$ is continuous such that for $\forall \overline{x} \in \widetilde{M}^n[T]$,

$$g \circ f(\overline{x}, 0) = \overline{x},$$
$$g \circ f(\overline{x}, 1) = p(\overline{x}),$$

and $g \circ f(v_0, t) = v_0$ for $\forall t \in I$, i.e., $\{v_0\}$ is a deformation retract of $\widetilde{M}^n[T]_1$. This completes the proof. □

2.2 Labeled Dimensional Graphs $\widetilde{M}^n[G^L]$

In Class 2, non-homotopic loops come from both $\varphi_\mu^{-1}(\mathbf{R}^n) \cap \varphi_\nu^{-1}(\mathbf{R})$ for $\mu, \nu \in \Lambda$ and the combinatorial structure G of $\mathscr{E}_G(n, \lambda \in \Lambda)$, which enables us to construct the labeled dimensional graph $\widetilde{M}^n[G^L]$ following.

Definition 2.8 *Let $\mathscr{A}[M^n] = \{\ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda\}$ be an atlas of an orientable manifold M^n. A labeled dimensional graph $\widetilde{M}^n[G^L]$ of a combinatorial Euclidean space $\mathscr{E}_G(n, \lambda \in \Lambda)$ is such a n-dimensional graph $\widetilde{M}^n[G]$ with labeling $L :\ (U_\mu, U_\nu) \to \kappa_{\mu\nu} + 1$ for $\forall (U_\mu, U_\nu) \in E(G)$, where $\kappa_{\mu\nu}$ is the number of non-homotopic loops between U_μ and U_ν for $\mu, \nu \in \Lambda$.*

Notice that such an edge labeled graph G^L can be simply viewed as a graph H_G with multiple edges by defining.

$V(H_G) = V(G^L)$;

$E(H_G) = \{(U_\mu, U_\nu)_i,\ 1 \le i \le \kappa_{\mu\nu} + 1 \mid \text{for } \forall (U_\mu, U_\nu) \in E(G),\ \mu, \nu \in \Lambda\}$.

Convention 2.9 *An edge labeled graph G^L with an integer labeling $L(e) \ge 1$ for $\forall e \in E(G)$ is viewed as a multiple graph H_G, i.e., $G^L = H_G$ throughout this paper.*

By this convention, such labeled graphs can be used essentially to characterize non-homotopic loops of manifolds. For example, the torus is a combinatorial Euclidean space $\mathscr{E}_G(2, \lambda = 1, 2)$ underlying a dipole $D_{0,3,0}$, i.e., consists of two Euclidean space $U_1,\ U_2 \cong \mathbf{R}^2$ with $\kappa_{ij} = 2$ non-homotopic loops and its labeled graph on $P_2^L = D_{0,3,0}$, such as those shown in Fig.2.2.

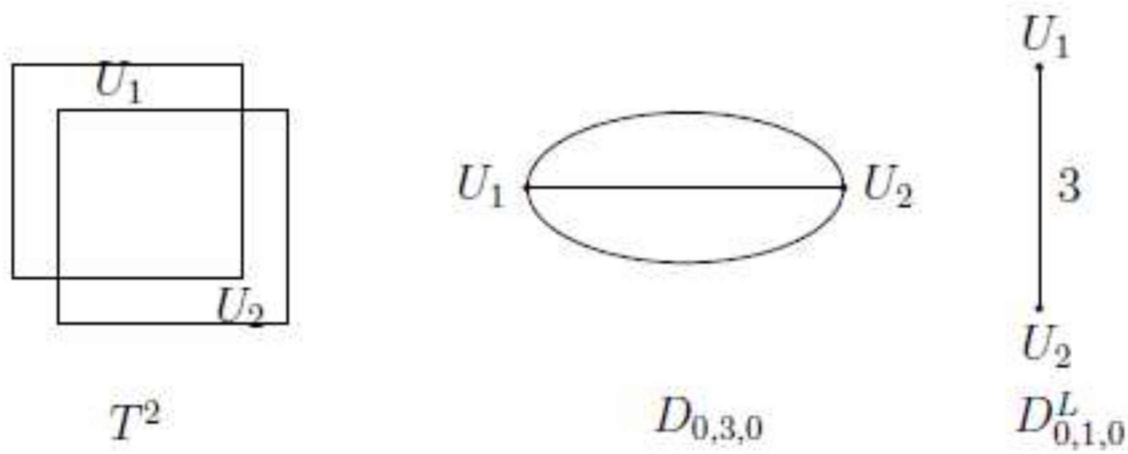

Fig.2.2

Convention 2.9 immediately implies the next result.

Theorem 2.10 *The number $\varpi(G^L)$ of cycles basis of an edge labeled graph G^L with an integer labeling $L(e) \ge 1$ for $\forall e \in E(G)$ is*

$$\varpi(G^L) = \varpi(G) + \sum_{L(e)\ge 2, e \in E(G)} (L(e) - 1).$$

Whence, $\varpi(G^L) = 0$ if and only if $\varpi(G) = 0$ and $L(e) = 1$ for $\forall e \in E(G)$.

Now denoted by $\mathscr{N}_C(\mathscr{E})$ and $\mathscr{N}_C(G^L)$ the sets of non-homotopic loops in the combinatorial Euclidean space $\mathscr{E}_G(n, \lambda \in \Lambda)$ and cycles in its labeled dimensional graph $\widetilde{M}^n[G^L]$, respectively. Then we know the following interesting result.

Theorem 2.11 *There is a bijection $\vartheta:\ \mathscr{N}_C(\mathscr{E}) \to \mathscr{N}_C(G^L)$, i.e.,*

$$\pi(\mathscr{E}_G(n, \lambda \in \Lambda))\ \cong\ \pi(\widetilde{M}^n[G^L]).$$

Proof We only need to prove that a loop $L \in \mathscr{N}_C(\mathscr{E})$ if and only if there is a cycle $C_L \in \mathscr{N}_C(G^L)$. The proof is divided into two cases following.

Case 1. L comes from the combinatorial structure G.

According to Theorems 2.1 and 2.4, we know that the underlying graphs of $\mathscr{E}_G(n, \lambda \in \Lambda)$ and $\widetilde{M}^n[G^L]$ are isomorphic. Whence there exists a cycle $C_L \in \mathscr{T}_0[G]$, i.e., $C_L \in \mathscr{N}_C(G^L)$ correspondent to L and verse via, for a cycle $C \in \mathscr{T}_0[G]$, there also exists a loop $L_C \in \mathscr{N}_C(\mathscr{E})$ for a cycle C. Whence, such a mapping $\vartheta: \ L \to C_L$ is a bijection.

Case 2. L comes from $U_\mu \cap U_\nu$ for two indexes $\mu, \nu \in \Lambda$.

Assume there are $\kappa_{\mu\nu}$ non-homotopic loos in $U \cap U_\nu$. Not loss of generality, let L be the ith loop. By Definition 2.8 and convention 2.9, we have a cycle C_L consisted of multiple edges $(U_\mu, U_\nu)_i$ with $(U_\mu, U_\nu)_{i+1}$ in the graph G^L. Then $\vartheta: \ L \to C_L$ is a bijection by definition.

Combining the discussion of Cases 1 and 2, we get a bijection

$$\vartheta: \ \mathscr{N}_C(\mathscr{E}) \to \mathscr{N}_C(H_G).$$

Notice that $\pi(\mathscr{E}_G(n)) = \langle \mathscr{N}_C(\mathscr{E}) \rangle$ and $\pi(\widetilde{M}^n[G^L]) = \pi(\widetilde{M}^n[H_G]) = \left\langle \mathscr{N}_C(G^L) \right\rangle$. The bijection $\vartheta: \ \mathscr{N}_C(\mathscr{E}) \to \mathscr{N}_C(G^L)$ naturally induces an isomorphism $\vartheta*: \ \pi(\mathscr{E}_G(n)) \to \pi(\widetilde{M}^n[G^L])$ by defining $\vartheta * (L_1 + L_2) = \vartheta(L_1) + \vartheta(L_2)$ for $L_1, \ L_2 \in \mathscr{N}_C(\mathscr{E})$ in the field $\mathbb{Z}_2$. Hence, we conclude that

$$\pi(\mathscr{E}_G(n)) \ \cong \ \pi(\widetilde{M}^n[G^L]). \qquad \square$$

Combining Theorems 2.1, 2.4 and 2.11, we get an important result on n-manifold with finite non-homotopic loops following.

Theorem 2.12 *For an n-manifold M^n with finite non-homotopic loops, there always exists a labeled n-dimensional graph $\widetilde{M}^n[G^L]$ such that*

$$\pi(M^n) \cong \pi(\widetilde{M}^n[G^L]).$$

Proof According to Theorem 2.1, there is a combinatorial Euclidean space $\mathscr{E}_G(n, \lambda \in \Lambda)$. Applying Theorems 2.4 and 2.11, we get a labeled n-dimensional graph $\widetilde{M}^n[G^L]$ such that

$$\pi(\mathscr{E}_G(n, \lambda \in \Lambda)) \cong \pi(\widetilde{M}^n[G^L]),$$

where G^L is defined in Definition 2.8. Whence, we know that

$$\pi(M^n) \cong \pi(\mathscr{E}_G(n, \lambda \in \Lambda)) \cong \pi(\widetilde{M}^n[G^L]).$$

This completes the proof. $\square$

Applying Theorems 2.4 and 2.12, we get more useful conclusions following.

Corollary 2.13 *The number of non-homotopic loops in an n-manifold M^n with finite non-homotopic loops is equal to the dimension of cycle space of its $\mathscr{T}_0[H_{G[M^n]}]$.*

Corollary 2.14 *For an integer $n \geq 2$, a compact n-manifold M^n is homotopy sphere if and only if $\widetilde{M}^n[G^L[M^n]]$ is a finite n-dimensional tree with labeling $L: \ e \rightarrow 1$ for $\forall e \in E(G^L[M^n])$.*

Proof By definition, M^n is compact if and only if $|G^L|$ is finite. Applying Theorems 2.10 and 2.12, we know that $\pi(M^n)$ is trivial if and only if $\pi(\widetilde{M}^n[G^L])$ is trivial, i.e., each labeling of edges is 1. Now by Corollary 2.5, there must be $\pi(\mathscr{T}_0[H_G], v_0) = \{v_0\}$ for $v_0 \in \mathscr{T}_0[H_G] \cap M^n$. But this can happens only if $\mathscr{T}_0[H_G]$ is a finite tree by Theorem 2.6. Whence, $\widetilde{M}^n[G^L[M^n]]$ is an n-dimensional tree with a labeling $L(e) = 1$ for $\forall e \in E(G^L[M^n])$. $\square$

Corollary 2.14 enables us to obtain an unified proof for the generalized Poincaré conjecture, i.e., *any homotopy n-sphere is homeomorphic to S^n* following.

[**Proof of Theorem** 1.4]

For any integer $n \geq 1$, by Corollary 2.14 an n-manifold M^n is homotopy n-sphere if and only if $\widetilde{M}[G^L[M^n]]$ defined in Definition 2.8 is a finite labeled n-dimensional tree with $L(e) = 1$ for $\forall e \in E(G)$, i.e., a finite n-dimensional tree. Applying Theorem 2.7, such an n-dimensional tree $\widetilde{M}[G^L[M^n]]$ can deformation retract to a point $v_0 \in M^n$. Whence, M^n is homeomorphic to an n-sphere S^n. $\square$

§3. Listing n-Manifolds by Labeled Graphs

Theorem 3.1 *Let $\mathscr{A}[M^n] = \{\ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda\}$ be a atlas of a locally compact n-manifold M^n. Then the dimensional graph $\widetilde{M}^n[G^L_{|\Lambda|}]$ of M^n is a topological invariant on $|\Lambda|$, i.e., if $\widetilde{M}^n[H^{L_1}_{|\Lambda|}]$ and $\widetilde{M}^n[G^{L_2}_{|\Lambda|}]$ are two labeled n-dimensional graphs of M^n, then there exists a self-homeomorphism $h: M^n \rightarrow M^n$ such that $h: H^{L_1}_{|\Lambda|} \rightarrow G^{L_2}_{|\Lambda|}$ naturally induces an isomorphism of graph.*

Proof Let

$$\mathscr{A}^1_{|\Lambda|}[M^n] = \{\ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda_1\}$$

and

$$\mathscr{A}^2_{|\Lambda|}[M^n] = \{\ (V_\lambda; \phi_\lambda) \mid \lambda \in \Lambda_2\}$$

be two minimum atlases of a locally compact n-manifold M^n with labeled graphs $H^{L_1}_{|\Lambda|}$ and $G^{L_2}_{|\Lambda|}$, respectively, where $\Lambda_1 = \Lambda_2 = \{1, 2, 3, \cdots, k, \cdots\}$ are countable index sets. Notice that $\varphi_\lambda : U_\lambda \rightarrow \mathbf{R}^n$ and $\phi_\lambda : V_\iota \rightarrow \mathbf{R}^n$ are homeomorphisms for $\forall \lambda \in \Lambda_1$

and $\iota \in \Lambda_2$. So there is a homeomorphism $\tau_\lambda : \varphi_\lambda(U_\lambda) \to \phi_\lambda(V_\lambda)$ for $\lambda \in \Lambda_1 = \Lambda_2$. Now define

$$h_\lambda = \phi_\lambda^{-1}\tau_\lambda\varphi_\lambda : x \to \phi_\lambda^{-1}(\tau_\lambda(\varphi_\lambda(x))) \ \text{ for } \ x \in U_\lambda.$$

Then h_λ with its inverse $h^{-1} = \varphi_\lambda^{-1}\tau_\lambda^{-1}\phi_\lambda$ is continuous on M^n. By the compactness of M^n there exists a partition of unity $c_\lambda : U_\lambda \to \mathbf{R}^n$, $\lambda \in \Lambda$ on the atlas $\mathscr{A}^1_{|\Lambda|}[M^n]$. Define a function $h : M^n \to M^n$ by

$$h = \sum_{\lambda\in\Lambda} \phi_\lambda^{-1}\tau_\lambda\varphi_\lambda c_\lambda,$$

where $A_\lambda = \operatorname{supp}(\varphi_\lambda)$ and

$$\phi_\lambda^{-1}\tau_\lambda\varphi_\lambda c_\lambda(x) = \begin{cases} c_\lambda(x)\phi_\lambda^{-1}\tau_\lambda\varphi_\lambda(x) & \text{if } x \in U_\lambda, \\ \mathbf{0} = (0, \cdots, 0) & \text{if } x \in U_\lambda - A_\lambda. \end{cases}$$

Then $h : M^n \to M^n$ is a homeomorphism with $h(U_\lambda) = V_\lambda$ for $\lambda \in \Lambda_1 = \Lambda_2$. Hence, $h : \ V(H^L_{|\Lambda|}) \to V(G^L_{|\Lambda|})$ is a bijection by definition.

Now if there are $\kappa_{\mu\nu}$ non-homotopic loops between U_μ and U_ν, then there are must be $\kappa_{\mu\nu}$ non-homotopic loops between V_μ and V_ν and vice via by the homeomorphic property. Therefore, $L_1 : (U_\mu, U_\nu) \to \kappa_{\mu\nu} + 1$ in $H^L_{|\Lambda|}$ if and only if $L_2 : (V_\mu, V_\nu) \to \kappa_{\mu\nu} + 1$ in $G^{L_2}_{|\Lambda|}$, i.e., $h(U_\mu, U_\nu) = (h(U_\mu), h(V_\nu))$ with $L_1(U_\mu, U_\nu) = L_2(h(U_\mu), h(U_\nu))$. By definition, two labeled graphs G^L_1 and G^L_2 with labeling mappings L_1 and L_2 are said to be isomorphic if there is an isomorphism $\varpi : \ G_1 \to G_2$ with $\varpi L_1 = L_2\varpi$. Whence, $h : \ H^L_{|\Lambda|} \to G^L_{|\Lambda|}$ naturally induces an isomorphism between labeled graphs $H^{L_1}_{|\Lambda|}$ and $G^{L_2}_{|\Lambda|}$. $\square$

We get a conclusion by Theorem 3.1 following.

Corollary 3.2 *The labeled graph $G^L_{|\Lambda|}$ of a locally compact n-manifold M^n is unique dependent on $|\Lambda|$.*

For classifying n-manifolds by applying Theorem 3.1, we introduce the conception of minimum atlas following.

Definition 3.3 *An atlas*

$$\mathscr{A}[M^n] = \{ \ (U_\lambda; \varphi_\lambda) \ | \ \lambda \in \Lambda\}$$

of an n-manifold M^n is minimal if there are no indexes $\mu, \ \nu \in \Lambda$ and a continuous mapping $\varphi_{\mu\nu}$ with $\varphi_{\mu\nu} : \ U_\mu \cup U_\nu \to \mathbf{R}^n$ such that

$$\mathscr{A}' = \{ \ (U_\lambda; \varphi_\lambda), (U_\mu \cup U_\nu, \varphi_{\mu\nu}) \ | \ \lambda \in \Lambda \setminus \{\mu, \nu\}\}$$

is also an atlas of M^n.

An atlas $\mathscr{A}[M^n]$ of an n-manifold M^n is minimum if it has minimum cardinality among all of its minimal atlases. Denoted such a minimal atlas by $\mathscr{A}_{min}[M^n]$ and its labeled n-dimensional graph by $\widetilde{M}^n[G^L_{min}]$.

The next result characterizes minimal atlases of an n-manifold.

Theorem 3.4 *Let*

$$\mathscr{A}[M^n] = \{ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda\}$$

be an atlas of a locally compact n-manifold M^n. Then

(i) $\mathscr{A}[M^n]$ is minimal if and only if there are no indexes $\mu,\ \nu \in \Lambda$ such that $U_\mu \cap U_\nu$ is arcwise connected, i.e., $L(e) \geq 2$ for $\forall e \in E(G^L_{min})$.

(ii) M^n is with finite non-homotopic loops if and only if $\mathscr{A}_{min}[M^n]$ is finite.

Proof (i) If the result (i) is not true, then there exist indexes $\mu,\ \nu \in \Lambda$ such that $U_\mu \cap U_\nu$ is arcwise connected. Assume

$$\varphi_\mu(U_\mu \cap U_\nu) = S \subset \mathbf{R}^n \ \text{ and } \ \varphi_\nu(U_\mu \cap U_\nu) = T \subset \mathbf{R}^n.$$

Notice that S and T are homeomorphic to $\mathbf{R}^n$ by definition. We can always choose a continuous mapping $\tau:\ S \to T$, i.e., $\tau(S) = T$. Define

$$\varphi^\tau_\mu = \tau\varphi_\mu.$$

Then we get that $\varphi^\tau_\mu|_{U_\mu\cap U_\nu} = \varphi_\nu|_{U_\mu\cap U_\nu}$. Whence, there is a continuous mapping $\varphi_{\mu\nu}:\ U_\mu \cup U_\nu \to \mathbf{R}^n$. Therefore,

$$\mathscr{A}' = \{ (U_\mu \cup U_\nu; \varphi_{\mu\nu}), (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda \setminus \{\mu,\nu\}\}$$

is also an atlas of M^n but with $|\mathscr{A}'| = |\Lambda| - 1$. This contradicts to the minimality of Λ. So (i) holds.

(ii) If $\mathscr{A}_{min}[M^n]$ is finite, then M^n is obvious only with finite non-homotopic loops by the assumption of its locally compactness. Now let

$$\mathscr{A}_{min}[M^n] = \{ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda\}$$

be a minimum atlas of a locally compact manifold M^n with an infinite index set Λ. By (i), there are no indexes μ,ν in Λ such that $U_\mu \cap U_\nu$ is arcwise connected. In other words, $U_\mu \cap U_\nu = \emptyset$ or with more than 2 arcwise components for $\forall \mu,\nu \in \Lambda$, i.e., $L(e) \geq 2$ for $\forall e \in E(G^L_{min})[M^n]$. Applying Theorem 2.10, we know that the number

$$\varpi(G^L_{min}[M^n]) = \varpi(G_{min}[M^n]) + \sum_{e \in E(G_{min}[M^n])} (L(e) - 1)$$

of cycle basis of $G^L_{min}[M^n]$ is greater than any sufficient larger number $N > 0$, which contradicts the assumption that M^n is only with finite non-homotopic loops. $\square$

Combining Corollary 3.2 with that of Theorem 3.4, we get a conclusion following, which enables us to list locally compact n-manifolds with finite non-homotopic loops by labeled graphs.

Corollary 3.5 *If the minimum labeled graph $G^L_{min}[M^n_1]$ of a locally compact n-manifold M^n_1 is not isomorphic to the minimum labeled graph $G^L_{min}[M^n_2]$ of M^n_2, then M^n_1 is not homeomorphic to M^n_2.*

Now by Theorem 3.4, let

$\mathscr{A}_{min}[M^n] = \{\ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda, |\Lambda| < +\infty\}$

be a minimum atlas of locally compact n-manifolds M^n. Then we can list n-manifolds following by Corollary 3.5.

(1) $|\Lambda| = 1$

In this case, $\mathscr{A}_{min}[M^n] = \{(U; \varphi)\}$, i.e., $M^n = \mathbf{R}^n$.

(2) $|\Lambda| = 2$

In this case,

$$\mathscr{A}_{min}[M^n] = \{(U_1; \varphi_1),\ (U_2; \varphi_2)\}$$

and M^n is double covered, which can be classified by labeled graphs $D^L_{0,1,0}$ shown in Fig.3.1,

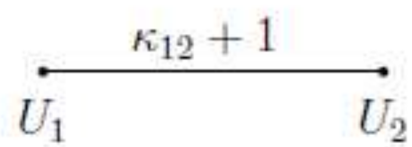

Fig.3.1

For example, if $n = 2$, i.e., compact 2-manifolds $\mathbf{S}$, then κ_{12} is the genus of $\mathbf{S}$ with $\kappa_{12} \geq 1$.

(3) $|\Lambda| = 3$

In this case,

$$\mathscr{A}_{min}[M^n] = \{(U_1; \varphi_1),\ (U_2; \varphi_2),\ (U_3; \varphi_3)\}$$

and we can list such n-manifolds M^n by labeled graphs in Fig.3.2, where integers $k, l, s \geq 2$.

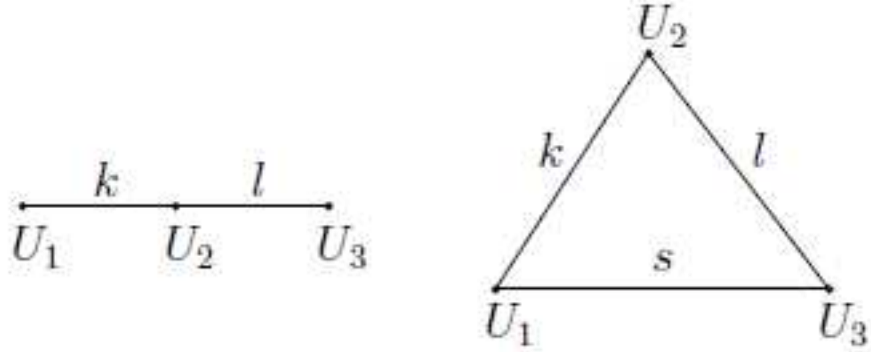

Fig.3.2

(4) $|\Lambda| = 4$

In this case,

$$\mathscr{A}_{min}[M^n] = \{(U_1;\varphi_1),\ (U_2;\varphi_2),\ (U_3;\varphi_3),\ (U_4;\varphi_4)\}$$

and we can list such n-manifolds M^n by labeled graphs in Fig.3.3, where integers $k, l, r, q, s, t \geq 2$.

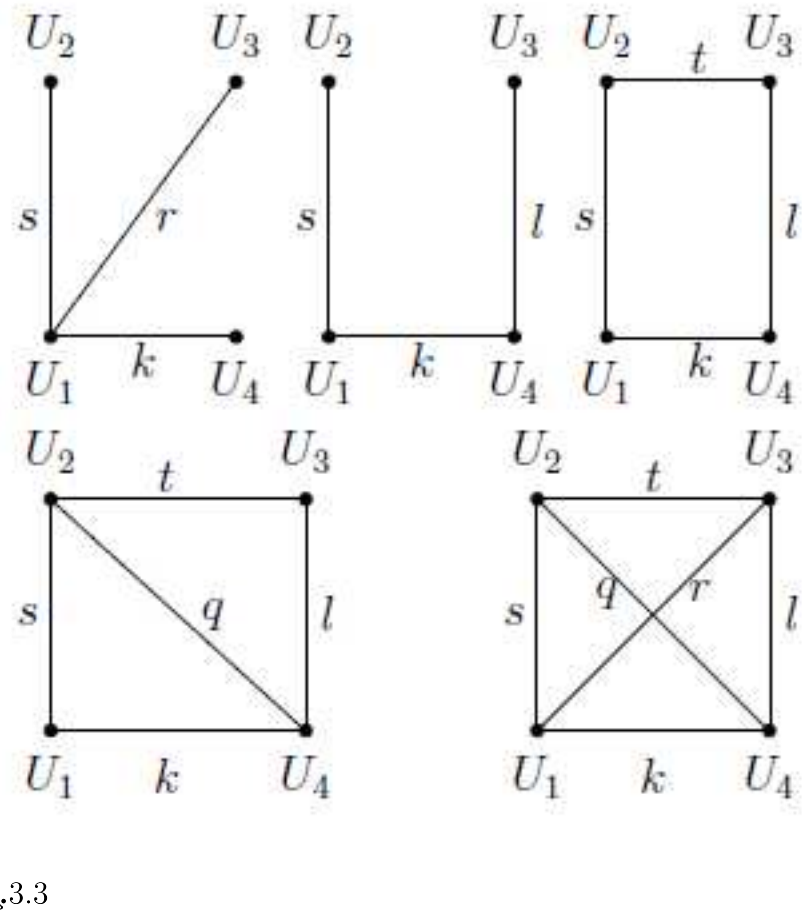

Fig.3.3

$\cdots\cdots\cdots\cdots$

(p) $|\Lambda| = p$

In this case,

$$\mathscr{A}_{min}[M^n] = \{(U_i; \varphi_i), 1 \leq i \leq p\}.$$

and there are $G_1, G_2, \cdots, G_{k(p)}$ such non-isomorphic graphs known in graph theory, where

$$k(p) \sim \frac{2^{\frac{p(p-1)}{2}}}{n!}$$

is the number of non-isomorphic graphs of order p with

p	1	2	3	4	5	6	7	8	9
$k(p)$	1	1	2	6	21	112	853	11117	261080

for $p \leq 9$. Then we can list such n-manifolds M^n by labeled graphs following:

$$G_1^L, G_2^L, \cdots, G_{k(p)}^L, \quad L_i(e_{ij}) \geq 2 \text{ for } \forall e_i \in E(G_i), 1 \leq i \leq k(p).$$

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